Commutative Algebra
Fall 2013 Lecture 13

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1 A Few Comments On Krull Dimension

Recall that R is a commutative ring. The Krull Dimension of R, if it exists, is the maximal height of any prime ideal of R.

Proposition: If $KdimR$ exists, then it is the largest height of any maximal ideal of R.

Proof: If P is nonmaximal but prime, then any chain ending at P can be extended by a max ideal Q containing P, so $height(P) < height(Q)$.

Proposition: Let R be an affine domain over F.
Then $KdimR = 0 \Leftrightarrow trdeg_F R = 0$.

Proof: $KdimR = 0$, a domain $\Leftrightarrow$ 0 is the only prime ideal $\Leftrightarrow$ R is a field $\Leftrightarrow$ R algebraic over F. (by Theorem A) $\Leftrightarrow trdeg_F R = 0$.

Proposition: Every maximal ideal of $F[\lambda_1, ..., \lambda_n]$ has height at least n and can be spanned by n elements as a module.

Proof (by induction on n):
For $n = 1$: $F[\lambda_1]$ is a PID, so all nonzero prime ideals are maximal and are generated by one element.
For $n > 1$: Let P be a maximal ideal of $R = F[\lambda_1, ..., \lambda_n]$, P contains a nonzero element of $F[\lambda_n]$, call it f. Let $K = F[\lambda_n]/ < f >$ in $F[\lambda_n]$. K is a field.

Let $P_n =< f > \subseteq P$, where $< f >$ is an ideal in R.

Consider P/P', $R/P' \cong K[\lambda_1, ..., \lambda_{n-1}]$ by the second isomorphism theorem. P/P' is a maximal ideal of R/P' so by induction P/P' has height at least $n-1$ and can be spanned by $n-1$ elements as a module.

Say $f_1 + P', f_2 + P', ..., f_{n-1} + P'$ span P/P'. So P is generated by $f_1, ..., f_{n-1}, f$. Suppose $Q_0 \subsetneq Q_1 \subsetneq ... \subsetneq Q_{n-1} = P/P'$ is a chain in $spec(R/P')$.

By the second isomorphism theorem $Q - i = P_i/P'$ and P_i is a prime ideal of R containing P'. So $P' \subseteq P_0 \subseteq P_1 \subseteq ... \subseteq P_{n-1} = P$. Finally, R is a domain so we can write $0 \subsetneq P' \subsetneq P_0 \subsetneq P_1 \subsetneq ... \subsetneq P_{n-1} = P$ giving that P has height $\geq n$.

2 LO, INC, GU

Running assumption: $C \subsetneq R$ commutative rings. By the third isomorphism theorem, if $Q \in Spec(R)$, then $C/(Q \cap C) \cong (C+Q)/Q$. Note that $(C+Q)/Q$ is a subring of R/Q but R/Q is an integral domain, so $(C+Q)/Q$ is also. So $Q \cap C \in Spec(C)$.

Definition: Call the map defined above
$\psi : Spec(R) \to Spec(C)$
$Q \mapsto Q \cap C$.
Say $Q \in Spec(R)$ lies over P in $Spec(C)$ if $P = Q \cap C$. Say $C \subseteq R$ satisfies the lying over condition (LO) if ψ is onto. In cases we care about, ψ is typically onto but not $1-1$.

Definition: Say $C \subseteq R$ satisfies the incomparibility condition (INC) if whenever $Q_0 \subsetneq Q_1$ in $spec(R)$ then $Q_0 \cap C \subsetneq Q_1 \cap C$.

The point is that if $Q_0 \neq Q_1 \in spec(R)$, both lying over $P \in Spec(C)$, then for an extension satisfying INC, we can't have $Q_0 \subsetneq Q$ or $Q_1 \subsetneq Q_0$, i.e prime ideals lying over P are incomparable in the subset partial order.

Definition: $C \subseteq R$ satisfies the going up condition (GU) if for all $P_0 \subseteq P_1 \in Spec(C)$ and for all $Q_0 \in Spec(R)$ lying over P_0.
$Q_0 \subseteq ?$
| |
P_0 - P_1
$\exists Q_1 \supseteq Q_0, Q_1 \in Spec(R)$. Q_1 lying over R.

Point:
1. Given $Q_0 \subsetneq Q_1 \subsetneq ... \subsetneq Q_m$ a chain in $Spec(R)$,
$Q_0 \cap C \subseteq Q_1 \cap C \subseteq ... \subseteq Q_m \cap C$ is a chain in $Spec(R)$, but it might have equalities. If $C \subseteq R$ satisfies INC, then there are no 'eq' in the second chain so the chains have the same length.
2. Given $P_0 \subsetneq P_1 \subsetneq ... \subsetneq P_m$ a chain in $Spec(C)$. If $C \subseteq R$ satisfies LO then $\exists Q_0 \in Spec(R)$ lying over P_0. If $C \subseteq R$ also satisfies GU, then inductively $\exists Q_1 \in Spec(R)$ for which
$Q_0 \subseteq Q_1 \subseteq Q_2 \subseteq ... \subseteq Q_M$
| | | |
$P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq ... \subsetneq P_m$
and the inclusions on the top chain are strict, since they are strict after inter-

secting with C. Thus both chains have the same length.

Proposition: If $C \subseteq R$ satisfies LO, INC, GU, then $KdimC = KdimR$.

3 LO, INC, GU for integral extensions

Lemma: Suppose Q is an ideal of R. 1. If R is integral over C, then R/Q is integral over $C/(C \cap Q)$. 2. If Q' is an ideal of R containing Q and $P' = Q' \cap C$. Then $P'/(Q \cap C) = Q'/Q \cap C/(C \cap Q)$.

Proof:
1. Take $r \in R$. Let f be a monic polynomial with coefficients in C, $f(r) = 0$. Then the image of f in R/Q has coefficients in $(C+Q)/Q \cong C/(C+Q)$. So the image of r is integral over $C/(C \cap Q)$.
2. Recall the modularity property of modules:
For $N_1 \subseteq N_2, K$ submodules of a module M, $(N_1 + K) \cap N_2 = N_1 + (K \cap N_2)$.
So we want $P'/(Q \cap C) = Q'/Q \cap C/(C \cap Q)$.
Equivalently, $((Q' \cap C) + Q)/Q = Q'/Q \cap (C+Q)/Q$. So it suffices to show $(Q' \cap C) + Q = Q' \cap (C + Q)$ which is true by modularity.

Lemma: Suppose R is an integral domain and R is algebraic over C. Then every nonzero ideal of R intersects C nontrivially.

Proof: Let $A \neq 0$ be an ideal of R and take $a \neq 0, a \in A$. Let $f = \sum_{i=1}^{n} c_i \lambda$ with $c_0 \neq 0$ and $f(a) = 0$. Then $c_0 = (-\sum_{i=1}^{n} c_0 a^{i-1})a \in A$. So $c_0 \in A \cap C$.

Proposition: Let $C \subseteq R$ be integral. Then INC holds.

Proof: Let $P \in Spec(C), Q_0 \subseteq Q_1 \in Spec(R)$ lying over P. Then $C/P \subseteq R/Q_0$ is also an integral extension, and both are integral domains. Furthermore, Q_1/Q_0 (an ideal of R/Q_0) lies over 0 in C/P, i.e. $Q_1/Q_0 \cap C/P = 0$ contradicting the lemma unless $Q_1/Q_0 = 0$, i.e. $Q_1 = Q_0$.

Lemma:
1. Let $S \subseteq R, S$ closed under multiplication, $1 \in S$. Then any ideal Q which is maximal wrt $Q \cap S = \phi$ is a prime ideal.
2. Suppose $P \in Spec(C)$ and A is an ideal of R with $A \cap C \subseteq P$. Then there exists an ideal Q containing A and maximal with respect to $Q \cap C \subseteq P$ and Q is necessarily a prime ideal.

Proof:
1. Check Q prime by checking that for any $B_1, B_2 \supsetneq Q$, B_1, B_2 ideals of R we have $B_1, B_2 \subsetneq Q$. Take $B_1, B_2 \supsetneq Q$ ideals of R. By hypothesis $B_1 \cap S \neq \phi$. Say $s_i \in B_i \cap S$. Then $s_1, s_2 \in B_1, B_2 \cap S$ so $s_1, s_2 \neq Q$ so $B_1 B_2 \subsetneq Q$.
2. Let $S = C \setminus P$. Let $\mathscr{S} = \{Q \text{ ideal of } R : Q \cap S = \phi, A \subseteq Q\}$. $\mathscr{S}$ is nonempty

as $A \in \mathscr{S}$. The union of any chain in $\mathscr{S}$ is in $\mathscr{S}$.
$\therefore$ by Zorn's Lemma, $\mathscr{S}$ contains a maximal element Q which is the Q we were looking for, and by 1, Q is prime.

Lemma: GU $\Rightarrow$ LO (for commutative rings)

Proof: Suppose $P \in Spec(C)$. By the Lemma with $A = 0, \exists Q_0 \in Spec(R)$ with $Q_0 \cap C \subseteq P$. Let $P_0 = Q_0 \cap C$. Then apply going up to get $Q \in Spec(R), Q_0 \subseteq Q, Q$ lying over P.

Proposition: Let $C \subseteq R$ integral. Then LO and GU hold.

Proof: By the previous lemma, we only need to prove GU. Take
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By 2 of the Lemma before, $\exists Q \supseteq Q_0$ in $Spec(R)$ maximal wrt $Q \cap C \subseteq P_1$.
Let $P = Q \cap C$. If $P = P_1$, we're done.
So assume $P \subsetneq P_1$. Take $a \in P_1 - P$.
Claim: $< Q, a > \cap C \subseteq P_1$.
This will suffice as it contradicts the maximality of Q. Take $r \in R$ st $ar \in C$.
By integrality $r^t = \sum_{i=0}^{t-1} c_i r^i$, $c_i \in C$, some t.
So $(ra)^t = a^i \sum_{i=0}^{t-1} c_i r^i = a \sum_{i=0}^{t-1} a^{t-i-1} c_i (ra)^i \in aC \subseteq P_1$.

If R is an integral extension of C, then $KdimR = KdimC$.